MEROMORPHIC CONTINUATION OF FUNCTIONS AND ARBITRARY DISTRIBUTION OF INTERPOLATION POINTS

M. BELLO HERNÁNDEZ AND B. DE LA CALLE YSERN

ABSTRACT. We characterize the region of meromorphic continuation of an analytic function f in terms of the geometric rate of convergence on a compact set of sequences of multi-point rational interpolants of f. The rational approximants have a bounded number of poles and the distribution of interpolation points is arbitrary.

1. INTRODUCTION

1.1. **Rational functions of best approximation.** It is well known that a sequence of polynomial or rational approximants converging on a certain region at geometric rate often converges in a larger region, giving thus additional information on the analytic or meromorphic continuation of the limit function. Walsh called that phenomenon overconvergence of the approximants (see, for instance, [16, 17]). Notice that the same word is used in a somewhat different setting to describe a property of power series (see [9], Chapter 11). One of the most beautiful theorems of this type was proved by Gonchar culminating a series of results given by Bernstein, Walsh, and Saff, among others.

Let K be a compact set of the complex domain $\mathbb{C}$ and G the unbounded component of $\overline{\mathbb{C}} \setminus K$. We say that K is regular if the domain G is regular with respect to the Dirichlet problem. By $\operatorname{cap}(K)$ we mean the logarithmic capacity of K. If $\operatorname{cap}(K) > 0$, there exists the generalized Green function of G with pole at $z = \infty$ which we denote by $g_G(z, \infty)$ (see [12], Section II.4). The fact that K is regular implies $\operatorname{cap}(K) > 0$.

Let $\mathcal{P}_n$ be the set of all complex polynomials of degree at most n. Set

$$\mathcal{R}_{n,m} = \{r \,:\, r = p/q,\ \ p \in \mathcal{P}_n,\ \ q \in \mathcal{P}_m,\ \ q \not\equiv 0\}$$

2010 *Mathematics Subject Classification.* Primary: 41A21; secondary: 41A25, 41A27.

Key words and phrases. Meromorphic continuation, overconvergence of rational approximants, row sequences of multipoint Padé approximants, arbitrary tables of interpolation.

This work was supported by Dirección General de Investigación, Ministerio de Educación y Ciencia, under grant MTM2009-14668-C02-02. The second author also received support from Universidad Politécnica de Madrid through Research Group "Constructive Approximation Theory and Applications".

and

$$d_{n,m}(K) = \min\{\|f - r\|_K \,:\, r \in \mathcal{R}_{n,m}\},$$

where $\|\cdot\|_K$ stands for the sup norm on K.

Theorem A. *Let f be a function defined on a regular compact set $K \subset \mathbb{C}$. Then,*

$$\limsup_{n\to\infty} \sqrt[n]{d_{n,m}(K)} \le 1/\theta < 1 \tag{1}$$

if and only if the function f admits meromorphic continuation with at most m poles on the set $E(\theta) = \{z \in \mathbb{C} \,:\, \exp\{g_G(z,\infty)\} < \theta\}$.

In particular, the best rational approximation characterizes the largest set $E(\theta)$ on which the function f admits meromorphic continuation with at most m poles. That is, if $\theta_{f,m}$ denotes the supremum of the numbers θ such that the function f admits meromorphic continuation with at most m poles on $E(\theta)$, then

$$\theta_{f,m} = 1/\limsup_{n\to\infty} \sqrt[n]{d_{n,m}(K)}.$$

Gonchar proved Theorem A in [2]. The essential contribution by Gonchar was to prove that (1) implies the assertion about the meromorphic continuation of f. The reciprocal statement basically follows from work by Walsh [17]. The first result in the direction of Theorem A was given by Bernstein [1] in 1912 for $m = 0$ and $K = [-1, 1]$. The general case for $m = 0$ was proved by Walsh [18]. Under the assumption that K is bounded by a Jordan curve, Saff proved Theorem A in [10].

Let f be an analytic function on a neighborhood of a compact set Σ wherein interpolation is carried out along an arbitrary table and let K be a regular compact set on which f is defined. Our main goal in this work is to show that we can characterize the largest region (of a given type) of meromorphic continuation of f by means of the geometric rate of convergence on K of its multi-point rational interpolants. Thus, we considerably enlarge both the class of situations and the regions in which we can deduce meromorphic continuation of f and show that the characterization of the largest region of meromorphic continuation does not depend on whether or not the interpolation table is extremal.

1.2. **Row sequences of Padé approximants.** To fix ideas, let us consider the simplest case of rational interpolants corresponding to classical Padé approximation. Let f be a function analytic on a neighborhood of $z = 0$ and let n and m be nonnegative integers. The Padé approximant of type (n, m) for f is defined as the unique rational function $\pi_{n,m} = p_{n,m}/q_{n,m}$ verifying

- $\deg p_{n,m} \le n$, $\deg q_{n,m} \le m$, and $q_{n,m} \not\equiv 0$.
- $q_{n,m}(z)f(z) - p_{n,m}(z) = A\, z^{n+m+1} + \cdots$,

where rational functions are identified if they coincide after cancellation of common factors from numerator and denominator. The problem of finding $\pi_{n,m}$ is reduced to that of solving a system of linear equations whose coefficients can be expressed in terms of the Taylor coefficients of f.

The table $\{\pi_{n,m}\}$, $n, m \in \mathbb{Z}_+$, is called the Padé table of f. For each $m \in \mathbb{Z}_+$ we say that D_m is the disk of m-meromorphy of f if D_m is the largest open disk with center at zero into which f can be continued as a meromorphic function that has at most m poles, counting multiplicities. The radius of D_m is denoted by R_m. The following result is implicitly contained in [4].

Theorem B. *Suppose that there exists $z \neq 0$ such that*

$$\limsup_{n\to\infty} |f(z) - \pi_{n,m}(z)|^{1/n} \leq \frac{|z|}{R} < 1. \tag{2}$$

Then, $R_m \geq R$, that is, f admits meromorphic continuation with at most m poles on the set $\{z \in \mathbb{C} : |z| < R\}$.

From (2) it follows that the disk of m-meromorphy D_m is characterized by the rate of convergence of the Padé approximants $\pi_{n,m}$ to f on a fixed point $z \neq 0$. The proof relies heavily on the fact that the table of interpolation points is newtonian (cf. Proposition 4.2 below). In the present paper we focus on the analogous problem for multi-point Padé approximants where the interpolation points tend to an arbitrary distribution and we prove that the phenomenon of overconvergence is not limited to approximants with maximal rate of convergence. This will be done in Section 4, see Theorem 4.1. Section 2 contains some definitions and auxiliary material whereas in Section 3 we characterize certain generalized sets of m-meromorphy of f in terms of convergence in σ-content (for the definition, see Section 2) of the multi-point Padé approximants. This implies that the region where the function f is proved to admit meromorphic continuation is the largest among those of a given type. In the case that the table of interpolation points is newtonian, we provide more detailed descriptions.

2. Definitions and auxiliary results

2.1. **Potential theory.** Let Σ be a compact set of the complex domain $\mathbb{C}$ with connected complement. Denote the domain $\overline{\mathbb{C}} \setminus \Sigma$ by Ω.

Let μ be a positive unit Borel measure supported on Σ. The logarithmic potential of μ is denoted by $P(\mu; z)$ and is equal to

$$\int_\Sigma -\log|z-\zeta|\, d\mu(\zeta).$$

Set

$$r_0 = \inf_{z\in\Sigma} \exp\{-P(\mu; z)\} \geq 0$$

and

$$E_\mu(r) = \{z \in \mathbb{C} : \exp\{-P(\mu; z)\} < r\}, \quad r > r_0.$$

Since the function $\exp\{-P(\mu;\cdot)\}$ is upper semi-continuous on $\mathbb{C}$, $E_\mu(r)$, $r > r_0$, is a non-empty bounded open set that, in general, does not contain the whole of Σ. By the same token $\exp\{-P(\mu;\cdot)\}$ attains its maximum on compact sets. So, for each compact set K let us denote

$$\rho_\mu(K) = \| \exp\{-P(\mu;\cdot)\}\|_K .$$

Lemma 2.1. *Given any open neighborhood U of Σ there exists $r_1 > r_0$ such that $E_\mu(r_1) \subset U$.*

Proof. First, notice that $\exp\{-P(\mu;z)\} \geq r_0$ for all $z \in \mathbb{C}$, due to the maximum principle for potentials (see Corollary II.3.3 in [12]).

We may assume that $F = \mathbb{C} \setminus U \neq \emptyset$. As $F \cap \Sigma = \emptyset$ and

$$-P(\mu;z) = \log|z| + o(1), \quad z \to \infty,$$

the function $\exp\{-P(\mu;\cdot)\}$ attains its minimum on F. Put

$$r_1 = \min_{z\in F} \exp\{-P(\mu;z)\} = \exp\{-P(\mu;z_1)\} \geq r_0, \quad z_1 \in F.$$

It is clear that $E_\mu(r_1) \subset U$. As $z_1 \notin \Sigma$, the function $P(\mu;\cdot)$ is harmonic on a neighborhood of z_1 on which cannot be constant because the complement of Σ has only one connected component. Therefore, by the minimum principle, there exists a point z_0 verifying $\exp\{-P(\mu;z_0)\} < r_1$, which implies that $r_1 > r_0$. □

If $\operatorname{cap}(\Sigma) > 0$ we have

$$g_\Omega(z,\infty) = -\log\operatorname{cap}(\Sigma) - P(\mu_\Sigma;z), \quad z \in \mathbb{C}, \tag{3}$$

where μ_Σ is the equilibrium measure of the set Σ.

Let μ_n and μ be finite positive Borel measures on $\overline{\mathbb{C}}$. By $\mu_n \xrightarrow{*} \mu$, $n \to \infty$, we denote the weak* convergence of μ_n to μ as n tends to infinity. This means that for every continuous function f on $\overline{\mathbb{C}}$ it holds

$$\lim_{n\to\infty} \int f(x)\, d\mu_n(x) \;=\; \int f(x)\, d\mu(x).$$

If all the measures μ_n, $n \in \mathbb{N}$, are supported on a fixed compact set F and $\mu_n \xrightarrow{*} \mu$, $n \to \infty$, the principle of descent for potentials (cf. [12], Theorem 1.6.8) says that

$$\liminf_{n\to\infty} P(\mu_n;z_n) \geq P(\mu;z_0), \tag{4}$$

where the sequence $\{z_n\}_{n\in\mathbb{N}}$ tends to $z_0 \in \mathbb{C}$ as n goes to infinity and

$$\lim_{n\to\infty} P(\mu_n;z) = P(\mu;z), \tag{5}$$

uniformly on compact subsets of $\mathbb{C} \setminus F$.

The fine topology on $\mathbb{C}$ is the coarsest topology on $\mathbb{C}$ for which all superharmonic functions are continuous. This is finer than ordinary planar topology. If D is a connected open set then the boundary of D in the fine and Euclidean topology coincide (see Theorem 5.7.9 in [6]).

2.2. **Convergence in σ-content.** Let A be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(A)$ we denote the class of all coverings of A by at most a numerable set of disks. Set

$$\sigma(A) = \inf\left\{\sum_{i\in I} |U_i| \,:\, \{U_i\}_{i\in I} \in \mathcal{U}(A)\right\},$$

where $|U_i|$ stands for the radius of the disk U_i. The quantity $\sigma(A)$ is called the 1-dimensional Hausdorff content of the set A. This set function is not a measure but fulfills some good properties like countable semiadditivity which, for instance, is not satisfied by the logarithmic capacity.

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on a domain D and φ another function also defined on D. We say that the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in σ-content to the function φ on compact subsets of D if for each compact subset K of D and for each $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \sigma\left(\{z \in K : |\varphi_n(z) - \varphi(z)| > \varepsilon\}\right) = 0.$$

Such a convergence will be denoted by $\sigma\text{-}\lim_{n\to\infty} \varphi_n = \varphi$ in D.

The next lemma was proved by Gonchar in [3].

Gonchar's Lemma. *Suppose that the sequence $\{\varphi_n\}$ of functions defined on a domain $D \subset \mathbb{C}$ converges in σ-content to a function φ on compact subsets of D. Then the following assertions hold true:*

i) *If each of the functions φ_n is meromorphic in D and has no more than $k < +\infty$ poles in this domain, then the limit function φ is also meromorphic (more precisely, it is equal to a meromorphic function in D except on a set of σ-content zero) and has no more than k poles in D.*

ii) *If each function φ_n is meromorphic and has no more than $k < +\infty$ poles in D and the function φ is meromorphic and has exactly k poles in D, then all φ_n, $n \geq N$, also have k poles in D; the poles of φ_n tend to the poles $z_1, , \ldots, z_k$ of φ (taking account of their orders) and the sequence $\{\varphi_n\}$ tends to φ uniformly on compact subsets of the domain $D' = D \setminus \{z_1, , \ldots, z_k\}$.*

2.3. **Multipoint Padé approximants.** Let f be an holomorphic function on a neighborhood V of the compact set $\Sigma \subset \mathbb{C}$. Let us fix a family of monic polynomials

$$w_n(z) = \prod_{i=1}^{n} (z - \alpha_{n,i}), \qquad n \in \mathbb{N}, \tag{6}$$

whose zeros are contained in Σ. It is easy to verify that for each pair of nonnegative integers n and m, $n \geq m$, there exists polynomials P and Q satisfying

- $\deg P \leq n - m$, $\deg Q \leq m$, and $Q \not\equiv 0$.

- $(Qf - P)/w_{n+1} \in \mathcal{H}(V)$, where $\mathcal{H}(V)$ denotes the space of analytic functions in V.

Any pair of such polynomials P and Q defines a unique rational function $\Pi_{n,m} = P/Q$ which is called the multi-point Padé approximant of type (n, m) of f.

By requiring $\deg P \leq n - m$ we have slightly modified the usual definition of a multi-point Padé approximant of type (n, m) in an equivalent form which is more suitable for the purposes of the paper. Let $\Pi_{n,m} = P_{n,m}/Q_{n,m}$ where $Q_{n,m}$ and $P_{n,m}$ are polynomials obtained eliminating all common zeros and, unless otherwise stated, normalizing $Q_{n,m}$ so that

$$Q_{n,m}(z) = \prod_{|\zeta_{n,k}| \leq 1} (z - \zeta_{n,k}) \prod_{|\zeta_{n,k}| > 1} \left(1 - \frac{z}{\zeta_{n,k}}\right). \tag{7}$$

Let $R_{\mu,m}$, $m \in \mathbb{Z}_+$, be the supremum of the numbers $r > r_0$ such that f admits meromorphic continuation with at most m poles on $E_\mu(r)$. Lemma 2.1 proves that $R_{\mu,m} > r_0$. We define the set of m-meromorphy of f relative to μ as the set $E_\mu(R_{\mu,m})$ and we denote it by $D_{\mu,m}$. It is easy to see that f admits meromorphic continuation with at most m poles on $D_{\mu,m}$.

For a given polynomial p, we denote by Θ_p the normalized zero counting measure of p. That is,

$$\Theta_p = \frac{1}{\deg p} \sum_{\xi\colon p(\xi)=0} \delta_\xi.$$

The sum is taken over all the zeros of p and δ_ξ denotes the Dirac measure concentrated at ξ. It is said that the sequence of interpolation points given by the polynomials $\{w_n\}_{n\in\mathbb{N}}$ has the measure μ as its asymptotic zero distribution if

$$\Theta_{w_n} \xrightarrow{*} \mu, \quad n \to \infty.$$

The following theorem is an analog of Montessus de Ballore's theorem (see [7]) for multi-point Padé approximants with arbitrary distribution of interpolation points. It was proved in [19] (cf. [15]) and constitutes a straightforward generalization of a previous result by Saff [11].

Theorem 2.2. *Let the measure μ be the asymptotic zero distribution of the sequence of interpolation points given by $\{w_n\}_{n\in\mathbb{N}}$. Suppose that f has exactly m poles on $D_{\mu,m}$. Then*

i) *For all $n \geq n_0$, $\Pi_{n,m}$ has exactly m poles which converge according to their multiplicity to the poles of the function f as n tends to infinity.*
ii) *For each compact subset $K \subset D_{\mu,m}$ not containing any pole of the function f, it holds*

$$\limsup_{n\to\infty} \|f - \Pi_{n,m}\|_K^{1/n} \leq \frac{\rho_\mu(K)}{R_{\mu,m}} < 1. \tag{8}$$

If, in the above theorem, we take $w_n(z) = z^n, n \in \mathbb{N}$, it follows that $\mu \equiv \delta_0$, $\Sigma = \{0\}$, $D_{\mu,m} = D_m$, and $R_{\mu,m} = R_m$; therefore regaining Montessus de Ballore's theorem. If we take Σ to be a regular compact set and μ precisely its equilibrium measure we obtain the same result as that of [11].

It is possible to prove an analog of (8) regardless whether or not the function f has exactly m poles on $D_{\mu,m}$. To this end, we need some additional definitions. Take an arbitrary $\varepsilon > 0$ and define the open set J_ε as follows. For $n \geq m$, let $J_{n,\varepsilon}$ denote the $\varepsilon/6mn^2$-neighborhood of the set of zeros of $Q_{n,m}$ and let $J_{m-1,\varepsilon}$ denote the $\varepsilon/6m$-neighborhood of the set of poles of f in $D_{\mu,m}(f)$. Set $J_\varepsilon = \cup_{n\geq m-1} J_{n,\varepsilon}$. We have $\sigma(J_\varepsilon) < \varepsilon$ and $J_{\varepsilon_1} \subset J_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$. For any set $B \subset \mathbb{C}$ we put $B(\varepsilon) = B \setminus J_\varepsilon$.

From these properties it readily follows that if $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly to the function φ on $K(\varepsilon)$ for every compact $K \subset D$ and for each $\varepsilon > 0$, then $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in σ-content to φ on compact subsets of D.

Due to the normalization (7), for any compact set K of $\mathbb{C}$ and for every $\varepsilon > 0$, there exist positive constants C_1, C_2, independent of n, such that

$$\|Q_{n,m}\|_K < C_1, \qquad \min_{z\in K(\varepsilon)} |Q_{n,m}(z)| > C_2 n^{-2m}, \tag{9}$$

where the second inequality is meaningful when $K(\varepsilon)$ is a non-empty set.

Lemma 2.3. *Let the measure μ be the asymptotic zero distribution of the sequence of interpolation points given by $\{w_n\}_{n\in\mathbb{N}}$. Then, for each compact set $K \subset D_{\mu,m}$ and for each $\varepsilon > 0$, it holds*

$$\limsup_{n\to\infty} \|f - \Pi_{n,m}\|_{K(\varepsilon)}^{1/n} \leq \frac{\rho_\mu(K)}{R_{\mu,m}} < 1. \tag{10}$$

Proof. Fix a compact set $K \subset D_{\mu,m}$ and $\varepsilon > 0$. Let $Q_m(z) = \prod_{i=1}^{\nu}(z - z_i)$, where z_i, $i = 1, \ldots, \nu \leq m$, are the poles of f on $D_{\mu,m}$. From the definition of multi-point Padé approximants it follows that the function

$$\frac{Q_m\, Q_{n,m}\, f - Q_m\, P_{n,m}}{w_{n+1}}$$

is holomorphic on $D_{\mu,m} \cup V$. Let $\eta > 0$ arbitrarily small. Let Γ be a cycle contained in

$$\{D_{\mu,m} \cup V\} \setminus \left\{\overline{E_\mu(R_{\mu,m} - \eta)} \cup \Sigma\right\}$$

homologous to 0 in $D_{\mu,m} \cup V$, with winding number equal to 1 for all the poles z_i, $i = 1, \ldots, \nu$, as well as the points of $K \cup \Sigma$. This can be done because the compact set $\overline{E_\mu(R_{\mu,m} - \eta)} \cup \Sigma$ is included in the open set $D_{\mu,m} \cup V$. Then, for all $z \in \Gamma$, we have

$$\exp\{-P(\mu; z)\} > R_{\mu,m} - \eta. \tag{11}$$

Applying the Cauchy integral formula, we obtain

$$\frac{[Q_m\, Q_{n,m}\, f - Q_m\, P_{n,m}](z)}{w_{n+1}(z)} = \frac{1}{2\pi i}\int_\Gamma \frac{[Q_m\, Q_{n,m}\, f](\zeta)}{w_{n+1}(\zeta)}\,\frac{d\zeta}{\zeta - z}$$

$$- \int_\Gamma \frac{[Q_m\, P_{n,m}](\zeta)}{w_{n+1}(\zeta)}\,\frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_\Gamma \frac{[Q_m\, Q_{n,m}\, f](\zeta)}{w_{n+1}(\zeta)}\,\frac{d\zeta}{\zeta - z},$$

for $z \in K$, where the second integral after the first equality is zero due to the fact that the integrand is an analytic function outside Γ with a zero of multiplicity at least two at infinity.

Then, for all $z \in K$

$$[Q_m\, Q_{n,m}\,(f - \Pi_{n,m})](z) = \frac{w_{n+1}(z)}{2\pi i}\int_\Gamma \frac{[Q_m\, Q_{n,m}\, f](\zeta)}{w_{n+1}(\zeta)}\,\frac{d\zeta}{\zeta - z}$$

and using (9) we arrive at

$$|f(z) - \Pi_{n,m}(z)| \le C(K,\eta,\varepsilon)\, n^{2m}\, \frac{\|w_{n+1}\|_K}{\min\limits_{\zeta\in\Gamma} |w_{n+1}(\zeta)|}, \tag{12}$$

for all $z \in K(\varepsilon)$, where the constant $C(K,\eta,\varepsilon)$ is independent of n.

As the measure μ is the asymptotic zero distribution of the sequence $\{w_n\}_{n\in\mathbb{N}}$, using (4) and (5), we have

$$\limsup_{n\to\infty} |w_n(z_n)|^{1/n} \le e^{-P(\mu;z_0)}, \tag{13}$$

where the sequence $\{z_n\}_{n\in\mathbb{N}}$ tends to $z_0 \in \mathbb{C}$ as n goes to infinity and

$$\lim_{n\to\infty} |w_n(z)|^{1/n} = e^{-P(\mu;z)}, \tag{14}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Sigma$.

It follows from (13) that

$$\limsup_{n\to\infty} \|w_{n+1}\|_K^{1/n} \le \rho_\mu(K). \tag{15}$$

Therefore, with the aid of (12), (14), (15), and (11), we obtain

$$\limsup_{n\to\infty} \|f - \Pi_{n,m}\|_{K(\varepsilon)}^{1/n} \le \frac{\rho_\mu(K)}{R_{\mu,m} - \eta}$$

and formula (10) follows from the above expression making η go to 0. □

Obviously, a proof of Theorem 2.2 may be obtained by using Lemma 2.3 and part ii) of Gonchar's lemma.

Let us denote by $R'_{\mu,m}$ the supremum of the numbers $r > r_0$ such that the sequence $\{\Pi_{n,m}\}_{n\ge m}$ converges in σ-content on compact subsets of $E_\mu(r)$. Due to Lemma 2.3, we have $0 < R_{\mu,m} \le R'_{\mu,m}$. On the other hand, from part i) of Gonchar's lemma it follows that f admits meromorhic continuation with at most m poles in the region where the sequence $\{\Pi_{n,m}\}_{n\ge m}$ converges

in σ-content. Therefore, it is clear that $R_{\mu,m} = R'_{\mu,m}$ and we could have used this property for an alternative definition of $R_{\mu,m}$.

Finally, using (3) for the compact set K, it is very easy to see that the following corollary is essentially equivalent to one of the implications in Theorem A.

Corollary 2.4. *Let K be a regular compact set and let $\{r_n\}_{n\geq m}$, $r_n \in \mathcal{R}_{n-m,m}$, be a sequence of rational functions satisfying*

$$\limsup_{n\to\infty} \sqrt[n]{\|f - r_n\|_K} \leq \frac{\operatorname{cap}(K)}{\theta} < 1.$$

Then

$$\sigma\text{-}\lim_{n\to\infty} r_n = f \quad in \quad E_{\mu_K}(\theta),$$

where μ_K is the equilibrium measure of K.

3. Convergence in σ-content of row sequences

Throughout the rest of the paper we maintain the notations introduced above, i.e., Σ is a compact set with connected complement Ω and f is an analytic function on a neighborhood V of Σ. For each nonnegative integers n, m, $n \geq m$, $\Pi_{n,m}$ stands for the multipoint Padé approximant of type (n, m) of f with interpolation points given by w_{n+1}. Such points belong to Σ. Given a positive unit Borel measure μ supported on Σ, recall that $\rho_\mu(K) = \|\exp\{-P(\mu;\cdot)\}\|_K$ and $D_{\mu,m}$ is the set of m-meromorphy of f relative to μ.

In this section we show that convergence in σ-content of the approximants $\{\Pi_{n,m}\}$ is not possible outside $D_{\mu,m}$ provided we are not in the set of interpolation Σ. The corresponding result for the classical case $\Sigma = \{0\}$ appears in [13].

Theorem 3.1. *Let the measure μ be the asymptotic zero distribution of the sequence of interpolation points given by $\{w_n\}_{n\in\mathbb{N}}$. Suppose that the sequence $\{\Pi_{n,m}\}_{n\geq m}$ converges in σ-content on compact subsets of a neighborhood of the point $z_0 \in \mathbb{C} \setminus \Sigma$. Then, $z_0 \in D_{\mu,m}$.*

Proof. We argue by contradiction and divide the proof into three parts. First, we reduce the result to finding a negative constant which uniformly bounds from above a sequence of subharmonic functions $\{h_n\}_{n\geq m}$. In the second part we study the boundary values of $h_n, n \geq m$, and obtain certain estimates for them which allow us to define a harmonic majorant of all subharmonic functions considered. This harmonic majorant has the desired properties and with that we conclude the proof in part three.

1. Suppose that $z_0 \notin D_{\mu,m}$. As $z_0 \in \mathbb{C} \setminus \Sigma$ and $P(\mu; z)$ is a harmonic function in that region, there exist $z_1 \in \mathbb{C} \setminus \Sigma$ and $\delta > 0$ such that $D(z_1, \delta) = \{z \in \mathbb{C} : |z - z_1| < \delta\} \subset \mathbb{C} \setminus \Sigma$, $\exp\{-P(\mu; z)\} > R_{\mu,m}$ for all $z \in \overline{D(z_1, \delta)}$,

and the sequence $\{\Pi_{n,m}\}_{n\geq m}$ converges in σ-content on compact subsets of a neighborhood of $\overline{D(z_1,\delta)}$. Due to this latter fact, for all $n \geq N_1$, it holds

$$\sigma\left\{z \in \overline{D(z_1,\delta)} \,:\, |\Pi_{n,m}(z) - \Pi_{n-1,m}(z)| \geq 1\right\} < \frac{\delta}{3}.$$

Hence, there exists $\delta_1 \in (\delta/3, \delta)$ such that

$$|\Pi_{n,m}(z) - \Pi_{n-1,m}(z)| < 1, \quad |z - z_1| = \delta_1, \quad n \geq N_1, \tag{16}$$

where we have used the facts that σ-content is not increased under circular projection and that the σ-content of a line segment is half of its length.

As $z_0 \notin D_{\mu,m}$, we know that $R_{\mu,m} < +\infty$. Recall that V is the neighborhood of Σ in which f is analytic. Set

$$s_n \equiv Q_{n,m}\, Q_{n-1,m}\,(\Pi_{n,m} - \Pi_{n-1,m}) \in \mathcal{P}_n, \quad n \geq m.$$

To reach a contradiction it is enough to prove that the functions

$$h_n(z) = \frac{1}{n}\log|s_n(z)| + P(\mu; z) + \log R_{\mu,m}, \quad n \geq m,$$

are uniformly bounded by a negative constant on a neighborhood U of the set

$$\{z \in \mathbb{C}\setminus V \,:\, \exp\{-P(\mu; z)\} = R_{\mu,m}\} \neq \emptyset.$$

In that case, there exist constants $q < 1$ and $C_3 > 0$ verifying

$$|s_n(z)| < C_3 q^n, \quad n \geq N, \quad z \in U. \tag{17}$$

Fix $\varepsilon > 0$ and K compact subset of U. Then, because of (9) and (17), we have

$$|\Pi_{n,m}(z) - \Pi_{n-1,m}(z)| = \left|\frac{s_n(z)}{Q_{n,m}(z)\, Q_{n-1,m}(z)}\right| \leq \frac{C_3}{C_2^2}\, q^n\, n^{4m},$$

for $n \geq N$ and for all $z \in K(\varepsilon)$. As $\sum_{n=1}^{\infty} q^n\, n^{4m} < +\infty$, the sequence $\{\Pi_{n,m}\}_{n\geq m}$ converges uniformly on $K(\varepsilon)$. Since $\varepsilon > 0$ and K are arbitrary, part i) of Gonchar's lemma proves that the sequence $\{\Pi_{n,m}\}_{n\geq m}$ converges in σ-content on compact subsets of U to a meromorphic continuation of f with at most m poles, which contradicts the definition of $R_{\mu,m}$.

2. As the functions s_n are analytic, from the maximum principle in $D(z_1,\delta_1)$, (9), and (16), it follows that there exists a constant $C_4 > 0$ such that

$$|s_n(z)| < C_4, \quad z \in \overline{D(z_1,\delta_1)}, \quad n \geq N_1.$$

We also have

$$P(\mu; z) + \log R_{\mu,m} < m < 0, \quad z \in \overline{D(z_1,\delta_1)}.$$

Then, there exists a negative constant α such that

$$h_n(z) < \alpha < 0, \quad z \in \overline{D(z_1,\delta_1)}, \quad n \geq N_2 \geq N_1. \tag{18}$$

On the other hand, let $Q_m(z) = \prod_{i=1}^{\nu}(z - z_i)$, where z_i, $i = 1,\ldots,\nu \leq m$, are the poles of f in $D_{\mu,m}$. Let W be a bounded open set with connected complement such that $\overline{W}$ is regular, $\Sigma \subset W \subset \overline{W} \subset V$, and $\overline{W} \cap \overline{D(z_1,\delta)} =$

$\emptyset$. The existence of W follows from the Hilbert lemniscate theorem (see Theorem 5.5.8 in [8]). Consequently, ∂W does not contain any pole of the function f in $D_{\mu,m}$ and the set $D = \overline{\mathbb{C}} \setminus (\overline{W} \cup \overline{D(z_1, \delta_1)})$ is connected.

Let $\eta > 0$ be arbitrarily small. Let Γ be a cycle contained in

$$\{D_{\mu,m} \cup V\} \setminus \left\{ \overline{E_\mu(R_{\mu,m} - \eta)} \cup \overline{W} \right\},$$

homologous to 0 in $D_{\mu,m} \cup V$ with winding number equal to 1 for all the poles z_i, $i = 1, \ldots, \nu$, as well as the points in $\overline{W}$. This can be done because the compact set $\overline{E_\mu(R_{\mu,m} - \eta)} \cup \overline{W}$ is included in the open set $D_{\mu,m} \cup V$. Now, we follow the arguments given in the proof of Lemma 2.3. For all $z \in \Gamma$ it holds (11) and we can apply the Cauchy integral formula and the principle of descent (14) in an analogous way to obtain

$$\limsup_{n\to\infty} |Q_{n,m}(z)\,(f(z) - \Pi_{n,m}(z))|^{1/n} \leq \frac{e^{-P(\mu;z)}}{R_{\mu,m}},$$

uniformly on ∂W since $\partial W \cap \Sigma = \emptyset$. Then, using (9) we have

$$\limsup_{n\to\infty} |s_n(z)|^{1/n} \leq \frac{e^{-P(\mu;z)}}{R_{\mu,m}}, \tag{19}$$

uniformly on ∂W. Therefore, for any given $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}, N_\varepsilon \geq N_2$, such that

$$h_n(z) \leq \varepsilon, \quad z \in \partial W, \quad n \geq N_\varepsilon. \tag{20}$$

3. Now, for each $\varepsilon > 0$, a harmonic function ψ_ε is constructed as the solution on $D = \overline{\mathbb{C}} \setminus (\overline{W} \cup \overline{D(z_1, \delta_1)})$ of the Dirichlet problem given by the boundary values

$$\psi_\varepsilon(z) = \begin{cases} \varepsilon, & \text{for } z \in \partial W, \\ \alpha, & \text{for } z \in \partial D(z_1, \delta_1). \end{cases}$$

Since the functions h_n, $n \in \mathbb{N}$, are subharmonic on D and taking account of (18) and (20), we see that ψ_ε is a harmonic majorant of h_n for all $n \geq N_\varepsilon$. Fix any compact set $K \subset D$. From the two-constant theorem (see [8], Theorem 4.3.7) it follows that there exist positive constants m_K and M_K depending only on K such that

$$\psi_\varepsilon(z) \leq \varepsilon\, m_K + \alpha\, M_K, \quad z \in K, \quad \varepsilon > 0.$$

Thus, for $\varepsilon > 0$ sufficiently small, the function ψ_ε is bounded from above on K by a negative constant. The same property is then satisfied by the functions h_n, $n \geq N_\varepsilon$. In particular, we can take K in such a way that it contains the open set U of part 1, which finishes the proof. □

Remark 1. Let $\Lambda = \{n_k\} \subset \mathbb{N}$ be a subsequence such that

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1 \tag{21}$$

and suppose that $\{\Pi_{n,m}\}_{n\in\Lambda}$ converges in σ-content on compact subsets of a neighborhood of the point $z_0 \in \mathbb{C} \setminus \Sigma$. Then, we can deduce that $z_0 \in D_{\mu,m}$

since all the arguments employed in the proof of Theorem 3.1 are valid without any modification except to obtain (19) where (21) is strongly used.

We say that the interpolation points $\alpha_{n,i}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, (see (6)) form a newtonian table if w_n is divisor of w_{n+1} for all $n \in \mathbb{N}$, that is, if $\alpha_{n,i} = \alpha_i, i = 1, \ldots, n,\ n \in \mathbb{N}$.

In the case that $\Sigma = \{0\}$ pointwise divergence of the Padé approximants outside $\overline{D}_m$ has been proved by Gonchar in [4]. For multi-point Padé approximants, this was obtained by H. Wallin [15] (excluding the set of interpolation Σ) provided that the interpolation points form a newtonian table and that the function f has exactly m poles in $D_{\mu,m}$. This latter condition plays an essential role since it assures the convergence of the denominators of the approximants which is strongly used in [15].

We next prove that the sequence $\{\Pi_{n,m}\}_{n\geq m}$ diverges pointwise outside the set $\overline{D}_{\mu,m} \cup \Sigma$ under the only assumption that the interpolation points form a newtonian table. To the best of our knowledge, proving such a result for general tables of interpolation is an open problem, even for $m = 0$. In the particular case that the function f is meromorphic with simple poles, an easy argument based on the descent principle and the residue theorem gives the desired divergence (cf. [14]).

Proposition 3.2. *Let the measure μ be the asymptotic zero distribution of the sequence of interpolation points given by $\{w_n\}_{n\in\mathbb{N}}$ which form a newtonian table. Then, the sequence $\{\Pi_{n,m}\}_{n\geq m}$ diverges pointwise in $\mathbb{C} \setminus \big(\overline{D}_{\mu,m} \cup \Sigma\big)$.*

Proof. We may assume that $R_{\mu,m} < +\infty$. Otherwise, there is nothing left to prove. From the definition of multi-point Padé approximants and the fact that the table of point is newtonian it follows that

$$\frac{Q_{n,m}f - P_{n,m}}{w_{n+1}} \in \mathcal{H}(V), \qquad \frac{Q_{n+1,m}f - P_{n+1,m}}{w_{n+1}} \in \mathcal{H}(V).$$

Therefore

$$\frac{Q_{n,m}P_{n+1,m} - Q_{n+1,m}P_{n,m}}{w_{n+1}} \in \mathcal{H}(V). \tag{22}$$

The numerator of the fraction in (22) is a polynomial of degree at most $n+1$ with zeros at the zeros of w_{n+1}. Necessarily

$$Q_{n,m}(z)P_{n+1,m}(z) - Q_{n+1,m}(z)P_{n,m}(z) = A_n w_{n+1}(z),$$

or, equivalently

$$\Pi_{n+1,m}(z) - \Pi_{n,m}(z) = \frac{A_n w_{n+1}(z)}{Q_{n,m}(z)Q_{n+1,m}(z)}. \tag{23}$$

Considering telescopic sums, it follows that the sequence $\{\Pi_{n,m}\}_{n\geq m}$ converges or diverges with the series

$$\sum_{n\geq n_0} \frac{A_n w_{n+1}(z)}{Q_{n,m}(z)Q_{n+1,m}(z)},$$

where n_0 is chosen conveniently so that $Q_{n_0,m}(z) \neq 0$ at the specific point under consideration. Set

$$\frac{1}{R^*} = \limsup_{n\to\infty} |A_n|^{1/n}.$$

Fix $\varepsilon > 0$ and an arbitrary compact set $K \subset E_\mu(R^*)$. By virtue of (9) and (13), we obtain

$$\limsup_{n\to\infty} \left| \frac{A_n w_{n+1}(z)}{Q_{n,m}(z)Q_{n+1,m}(z)} \right|^{1/n} \leq \frac{\rho_\mu(K)}{R^*} < 1, \quad z \in K(\varepsilon),$$

which implies that the sequence $\{\Pi_{n,m}\}_{n\geq m}$ converges uniformly on $K(\varepsilon)$. Then $\{\Pi_{n,m}\}_{n\geq m}$ converges in σ-content on compact subsets of $E_\mu(R^*)$ and, consequently, $R^* \leq R_{\mu,m}$.

Let us consider $z_0 \notin \Sigma$ such that $\rho_\mu(\{z_0\}) > R^*$. Using now (9) and (14) we have

$$\limsup_{n\to\infty} \left| \frac{A_n w_{n+1}(z_0)}{Q_{n,m}(z_0)Q_{n+1,m}(z_0)} \right|^{1/n} \geq \frac{\rho_\mu(\{z_0\})}{R^*} > 1. \tag{24}$$

Hence, the sequence $\{\Pi_{n,m}(z)\}_{n\geq m}$ diverges at $z = z_0$.

It remains to prove that $R^* \geq R_{\mu,m}$. Suppose that $R^* < R_{\mu,m}$. Then, there exists $z_1 \in D_{\mu,m} \setminus \Sigma$ with $\rho_\mu(\{z_1\}) > R^*$. By the continuity of the potential $P(\mu;\cdot)$ in a neighborhood of z_1, there exists a compact disk $K_1 = \overline{D(z_1,\delta)} \subset D_{\mu,m} \setminus \Sigma$ verifying $\rho_\mu(\{z\}) > R^*$ for all $z \in K_1$.

Fix $\varepsilon > 0$ with $\varepsilon < \delta$ so that $K_1(\varepsilon) \neq \emptyset$. From (10) and (23) it follows

$$\limsup_{n\to\infty} \left| \frac{A_n w_{n+1}(z)}{Q_{n,m}(z)Q_{n+1,m}(z)} \right|^{1/n} \leq \frac{\rho_\mu(K)}{R_{\mu,m}} < 1, \quad z \in K_1(\varepsilon),$$

which is absurd in light of (24). $\square$

Remark 2. Notice that the proof of Proposition 3.2 does not make use of the fact that the set Σ is simply connected. Therefore, that condition may be dropped in this case. Instead, we must require the function f to be analytic in $E_\mu(r)$ for some $r > r_0$ since now we cannot apply Lemma 2.1. The same can be said about the other results concerning newtonian tables of interpolation, that is, Proposition 4.2 and Corollary 4.4 below.

4. Meromorphic continuation

We are ready for our main result.

Theorem 4.1. *Let the measure μ be the asymptotic zero distribution of the sequence of interpolation points given by $\{w_n\}_{n\in\mathbb{N}}$. Let K be a regular compact set for which the value $\rho_\mu(K)$ is attained at a point that does not belong to the interior of Σ. Suppose that the function f is defined on K and fulfills*

$$\limsup_{n\to\infty} \|f - \Pi_{n,m}\|_K^{1/n} \leq \frac{\rho_\mu(K)}{R} < 1. \tag{25}$$

Then, $R_{\mu,m} \geq R$, that is, f admits meromorphic continuation with at most m poles on the set $E_\mu(R)$.

Proof. Several arguments that will be used below are similar to those employed in the proof of Theorem 3.1. In particular, we divide the proof into three parts along the same lines.

1. Let us suppose that $R_{\mu,m} < R$, hence we will come into contradiction. It follows from (25), using Corollary 2.4, that the multi-point approximants $\Pi_{n,m}$ converge in σ-content to f on compact subsets of $E_{\mu_K}(\lambda_0)$, where

$$\lambda_0 = \mathrm{cap}\,(K)\,\frac{R}{\rho_\mu(K)} > \mathrm{cap}\,(K). \tag{26}$$

In case that $\rho_\mu(K) = 0$ or $R = +\infty$ we have $\lambda_0 = +\infty$ and Theorem 4.1 trivially holds true. Besides, from Theorem 3.1, we have $E_{\mu_K}(\lambda_0) \subset D_{\mu,m} \cup \Sigma$.

Let V be the neighborhood of Σ in which f is analytic and set

$$s_n \equiv Q_{n,m}\, Q_{n-1,m}\,(\Pi_{n,m} - \Pi_{n-1,m}) \in \mathcal{P}_n, \quad n \geq m.$$

Using the same argument that appears in the proof of Theorem 3.1, we see that in order to reach a contradiction it is enough to prove that the functions

$$h_n(z) = \frac{1}{n}\log|s_n(z)| + P(\mu; z) + \log R_{\mu,m}$$

are uniformly bounded by a negative constant on a neighborhood U of the set

$$\{z \in \mathbb{C} \setminus V \,:\, \exp\{-P(\mu; z)\} = R_{\mu,m}\} \neq \emptyset.$$

2. For technical reasons that will become apparent below, we fix $\lambda > \mathrm{cap}\,(K)$, $\lambda < \lambda_0$, sufficiently close to $\mathrm{cap}\,(K)$ so that

$$2\,\log\left(\frac{\lambda}{\mathrm{cap}\,K}\right) < \frac{1}{4}\log\left(\frac{R}{R_{\mu,m}}\right). \tag{27}$$

From (25) and (9) it follows that

$$\limsup_{n\to\infty} \|s_n\|_K^{1/n} \leq \frac{\mathrm{cap}\,(K)}{\lambda_0}. \tag{28}$$

Using the Bernstein-Walsh lemma (see Theorem 5.5.7 in [8]), it holds

$$\limsup_{n\to\infty} \|s_n\|_{F_\kappa}^{1/n} \leq \frac{\kappa}{\lambda_0} < 1, \tag{29}$$

for all κ with $\mathrm{cap}\,(K) < \kappa < \lambda_0$, where $F_\kappa = \overline{E_{\mu_K}(\kappa)}$.

Fix $\delta > 0$ such that

$$\delta < \log\left(\frac{\lambda}{\mathrm{cap}\,(K)}\right) \tag{30}$$

and $\epsilon > 0$ with $\epsilon < \lambda_0 - \lambda$ and $\log(\lambda + \epsilon) < \log\lambda + \delta/2$.

Apply (29) for $\kappa = \lambda + \epsilon$. Then, we have

$$\frac{1}{n}\log|s_n(z)| \leq \log(\lambda+\epsilon) - \log\lambda_0 + \delta/2, \quad n \geq n_1(\delta), \quad z \in E_{\mu_K}(\lambda+\epsilon).$$

Therefore,

$$\frac{1}{n}\log|s_n(z)| \le \log\lambda - \log\lambda_0 + \delta, \quad n \ge n_1(\delta), \quad z \in E_{\mu_K}(\lambda+\epsilon).$$

That is,

$$h_n(z) \le \log\lambda - \log\lambda_0 + P(\mu;z) + \log R_{\mu,m} + \delta, \tag{31}$$

for all $z \in E_{\mu_K}(\lambda+\epsilon)$ and $n \ge n_1(\delta)$.

Recall that $\Omega = \overline{\mathbb{C}} \setminus \Sigma$ is a connected open set. Let $z_0 \in K$ such that $P(\mu, z_0) = -\log\rho_\mu(K)$. From hypotheses, we may assume that $z_0 \in \overline{\Omega}$ which implies that z_0 is a limit point of Ω in the fine topology. As $K \subset E_{\mu_K}(\lambda+\epsilon)$, there exist an open disk B_0 centered at $z = z_0$ and a point $z_1 \in \Omega$ such that $z_1 \in B_0 \subset \overline{B}_0 \subset E_{\mu_K}(\lambda+\epsilon)$ and

$$P(\mu;z_1) < -\log\rho_\mu(K) + \frac{1}{8}\log\left(\frac{R}{R_{\mu,m}}\right).$$

Then, there exists an open disk B_1 centered at $z = z_1$ with $\overline{B}_1 \subset B_0 \cap \Omega$ and verifying

$$P(\mu;z) < -\log\rho_\mu(K) + \frac{1}{4}\log\left(\frac{R}{R_{\mu,m}}\right), \quad \text{for all} \quad z \in \overline{B}_1, \tag{32}$$

since $P(\mu;\cdot)$ is a continuous function on a neighborhood of z_1.

Therefore, using (31), (30), (26), (32), and (27), for all $z \in \overline{B}_1$ and $n \ge n_1(\delta)$, it holds

$$\begin{aligned}
h_n(z) &\le \log\lambda - \log\lambda_0 + P(\mu;z) + \log R_{\mu,m} + \delta \\
&= \log\left(\frac{\lambda}{\operatorname{cap}(K)}\right) + P(\mu;z) + \log R_{\mu,m} + \log\left(\frac{\operatorname{cap}(K)}{\lambda_0}\right) + \delta \\
&< 2\,\log\left(\frac{\lambda}{\operatorname{cap}(K)}\right) + P(\mu;z) + \log R_{\mu,m} + \log\left(\frac{\operatorname{cap}(K)}{\lambda_0}\right) \\
&= 2\,\log\left(\frac{\lambda}{\operatorname{cap}(K)}\right) + P(\mu;z) + \log\rho_\mu(K) - \log\left(\frac{R}{R_{\mu,m}}\right) \\
&< 2\,\log\left(\frac{\lambda}{\operatorname{cap}(K)}\right) - \frac{3}{4}\log\left(\frac{R}{R_{\mu,m}}\right) < -\frac{1}{2}\,\log\left(\frac{R}{R_{\mu,m}}\right).
\end{aligned}$$

Then, we obtain

$$h_n(z) < \alpha < 0, \quad \text{for all } z \in \overline{B}_1 \text{ and } n \ge n_1(\delta). \tag{33}$$

On the other hand, let $Q_m(z) = \prod_{i=1}^{\nu}(z - z_i)$, where z_i, $i = 1, \dots, \nu \le m$, are the poles of f in $D_{\mu,m}$. Let W be a bounded open set with connected complement such that $\overline{W}$ is regular, $\Sigma \subset W \subset \overline{W} \subset V$, and $\overline{W} \cap \overline{B}_1 = \emptyset$. Consequently, ∂W does not contain any pole of the function f in $D_{\mu,m}$ and the set $D = \overline{\mathbb{C}} \setminus (\overline{W} \cup \overline{B}_1)$ is connected.

Reasoning as in the proof of Theorem 3.1, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log |Q_{n,m}(z)\,(f-\Pi_{n,m})(z)| \le -P(\mu;z) - \log R_{\mu,m},$$

uniformly on ∂W. Hence, on account of (9), it holds

$$\limsup_{n\to\infty} h_n(z) \le 0, \tag{34}$$

uniformly on ∂W. Then, fixed $\varepsilon > 0$, we obtain

$$h_n(z) \le \varepsilon, \quad n \ge n_2(\varepsilon), \quad z \in \partial W. \tag{35}$$

3. For each $\varepsilon \ge 0$, a harmonic function ψ_ε is constructed as the solution on $D = \overline{\mathbb{C}} \setminus (\overline{W} \cup \overline{B}_1)$ of the Dirichlet problem given by the boundary values

$$\psi_\varepsilon(z) = \begin{cases} \varepsilon, & \text{for } z \in \partial W, \\ \alpha, & \text{for } z \in \partial B_1. \end{cases}$$

As the functions h_n are subharmonic on $D \subset \overline{\mathbb{C}} \setminus \Sigma$ and taking account of (35) and (33), we see that ψ_ε is a harmonic majorant of h_n for all $n \ge N = \max\{n_1(\delta), n_2(\varepsilon)\}$. Besides, the open set U of part 1 verifies $U \subset D$ since $E_{\mu_K}(\lambda_0) \setminus \Sigma \subset D_{\mu,m}$. The rest of the proof is analogous to that of part 3 of Theorem 3.1. □

Theorem 4.1 was proved in [5] for the particular case $m = 0$ under the additional assumption that the sets K and Σ coincide.

Remark 3. Due to the same reasons as in Remark 1, Theorem 4.1 may be proved under the slightly weaker hypothesis that formula (25) holds true only for a subsequence $\Lambda = \{n_k\} \subset \mathbb{N}$ verifying (21). Condition (21) is used here to obtain (28) and (34).

In the case that the table of interpolation points is newtonian, we can be more precise.

Proposition 4.2. *Let the measure μ be the asymptotic zero distribution of the sequence of interpolation points given by $\{w_n\}_{n\in\mathbb{N}}$ which form a newtonian table. Suppose that the function f is defined at a point $z \notin \Sigma$ and fulfills*

$$\limsup_{n\to\infty} |f(z) - \Pi_{n,m}(z)|^{1/n} \le \frac{\rho_\mu(\{z\})}{R} < 1.$$

Then, $R_{\mu,m} \ge R$, that is, f admits meromorphic continuation with at most m poles on the set $E_\mu(R)$.

Proof. From (23) it follows that

$$|A_n| \le \left| \frac{Q_{n,m}(z)\, Q_{n+1,m}(z)}{w_{n+1}(z)} \right| \left(|f(z) - \Pi_{n+1,m}(z)| + |f(z) - \Pi_{n,m}(z)|\right) \tag{36}$$

Hence, taking limits in (36) and using (9) and (14), we obtain

$$\limsup_{n\to\infty} |A_n|^{1/n} \le \frac{1}{\rho_\mu(\{z\})} \limsup_{n\to\infty} |f(z) - \Pi_{n,m}(z)|^{1/n}. \tag{37}$$

Notice that $\rho_\mu(\{z\}) > 0$ since $z \notin \Sigma$. From (37) and the proof of Proposition 3.2 it follows that

$$\frac{1}{R_{\mu,m}} = \limsup_{n\to\infty} |A_n|^{1/n} \leq \frac{1}{R},$$

which gives the result. □

The following corollary is a straightforward consequence of Theorem 4.1 and the definition of $R_{\mu,m}$.

Corollary 4.3. *Let the measure μ be the asymptotic zero distribution of the sequence of interpolation points given by $\{w_n\}_{n\in\mathbb{N}}$. Suppose that f has exactly m poles on $D_{\mu,m}$. Let $K \subset D_{\mu,m}$ be a regular compact subset not containing any pole of the function f and for which the value $\rho_\mu(K)$ is attained at a point that does not belong to the interior of Σ. Then, it holds*

$$\limsup_{n\to\infty} \|f - \Pi_{n,m}\|_K^{1/n} = \frac{\rho_\mu(K)}{R_{\mu,m}} < 1.$$

If we use Proposition 4.2 instead, we analogously obtain the next result.

Corollary 4.4. *Let the measure μ be the asymptotic zero distribution of the sequence of interpolation points given by $\{w_n\}_{n\in\mathbb{N}}$ which form a newtonian table. Suppose that f has exactly m poles z_i, $i = 1, \ldots, m$, on $D_{\mu,m}$. Then,*

$$\limsup_{n\to\infty} |f(z) - \Pi_{n,m}(z)|^{1/n} = \frac{\rho_\mu(\{z\})}{R_{\mu,m}} < 1,$$

for each $z \in D_{\mu,m} \setminus (\Sigma \cup \{z_1, \ldots, z_m\})$.

References

[1] S.N. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*, Mem. Acad. Roy. Belg. (2), **4** (1912), 1–104.

[2] A.A. Gonchar, *On a theorem of Saff*, Sb. Math., **23** (1974), 149–154.

[3] A.A. Gonchar, *On the convergence of generalized Padé approximants of meromorphic functions*, Sb. Math., **27** (1975), 503–514.

[4] A.A. Gonchar, *Poles of rows of the Padé table and meromorphic continuation of functions*, Sb. Math, **43** (1982), 527–546.

[5] R. Grothmann, *Distribution of interpolation points*, Ark. Mat., **34** (1996), 103–117.

[6] L.L. Helms, *Potential Theory*, Springer-Verlag, London 2009.

[7] R. de Montessus de Ballore, *Sur les fractions continues algébriques*, Bull. Soc. Math. France, **30** (1902), 28–36.

[8] T. Ransford, *Potential Theory in the Complex Plane*, London Mathematical Society, Student Texts 28, Cambridge University Press, New York 1995.

[9] R. Remmert, *Classical Topics in Complex Function Theory*, Springer, New York 1998.

[10] E.B. Saff, *Regions of meromorphy determined by the degree of best rational approximation*, Proc. Amer. Math. Soc., **29** (1971), 30–38.

[11] E.B. Saff, *An extension of Montessus de Ballore's theorem on the convergence of interpolating rational functions*, J. Approx. Theory **6** (1972), 63–67.

[12] E.B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, A Series of Comprehensive Studies in Math. 316, Springer-Verlag, Berlin 1997.

[13] V.V. Vavilov, *On the convergence of the Padé approximants of meromorphic functions*, Sb. Math., **30** (1976), 39–49.
[14] H. Wallin, *Divergence of multipoint Padé approximation*, Lecture Notes in Math. 1014, Springer, Berlin 1983, pp. 246–255.
[15] H. Wallin, *Convergence and divergence of multipoint Padé approximants of meromorphic functions*, Lecture Notes in Math. 1105, Springer, New York 1984, pp. 272–284.
[16] J.L. Walsh, *On the overconvergence of certain sequences of rational functions of best approximation*, Acta Math., **57** (1931), 411–435.
[17] J.L. Walsh, *The convergence of sequences of rational functions of best approximation*, Math. Annalen, **155** (1964), 252–264.
[18] J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*. Coll. Publ. XX, Amer. Math. Soc., Providence, Rhode Island 1969.
[19] D.D. Warner, *An extension of Saff's theorem on the convergence of interpolating rational functions*, J. Approx. Theory, **18** (1976), 108–118.

Departamento de Matemáticas y Computación, Universidad de La Rioja, c/ Luis de Ulloa s/n, 26004 Logroño, La Rioja, Spain.
E-mail address: mbello@unirioja.es

Departamento de Matemática Aplicada, E.T.S. Ingenieros Industriales, Universidad Politécnica de Madrid, José G. Abascal 2, 28006 Madrid, Spain.
E-mail address: bcalle@etsii.upm.es